

FOURIER-MUKAI TRANSFORMATIONS ON K3 SURFACES WITH $\rho = 1$ AND ATKIN-LEHNER INVOLUTIONS

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ABSTRACT. We show that there is a surjection from the Fourier-Mukai transformations on projective K3 surfaces with the Picard number $\rho(X) = 1$ to so called to the group of Atkin-Lehner involutions. This was expected in Hosono-Lian-Oguiso-Yau's paper.

1. INTRODUCTION

1.1. Terminologies and backgrounds. Let $D(M)$ be the bounded derived category of coherent sheaves on a projective manifold M . In this article a projective manifold M' is said to be a *Fourier-Mukai partner* of M if there is an equivalence $\Phi: D(M') \rightarrow D(M)$. Any equivalence $\Phi: D(M_1) \rightarrow D(M_2)$ between Fourier-Mukai partners of M is said to be a *Fourier-Mukai transformation* on M . The number of isomorphic classes of Fourier-Mukai partners of M is said to be the *Fourier-Mukai number* of M . It is conjectured that the Fourier-Mukai number of any projective manifold is finite by Kawamata in [Kaw02]. For instance the conjecture holds for curves (For example see [Huy, mainly in Chapter 5]) and surfaces ([BM01] and [Kaw02]). And also, the conjecture holds for abelian varieties (essentially [Orl02] and independently [Fav12]).

1.2. The study of [HLOYb]. The main interest of this paper is the relation, which is predicted by [HLOYb, Remark in page 25], between Atkin-Lehner involutions and the Fourier-Mukai number of projective K3 surfaces X with $\rho(X) = 1$. In the following we briefly recall the study of [HLOYb].

Suppose that X is a projective K3 surface with $\mathrm{NS}(X) = \mathbb{Z}L_X$ and with $L_X^2 = 2d$. The numerical Grothendieck group $\mathcal{N}(X)$ of X has the Mukai (or Euler) paring $\langle -, - \rangle$ with the signature $(2, 1)$. Then as was shown by Dolgachev, the isometry group of $O^+(\mathcal{N}(X))/\pm \mathrm{id}$ is isomorphic to Atkin-Lehner modular group AL_d of level d (See also Definition 2.3).

Now recall that any autoequivalence on $D(X)$ induces an isometry on $\mathcal{N}(X)$. Then we have a representation from $\mathrm{Aut}(D(X))$ to AL_d . By virtue of [HMS09, Corollary 3] we see the image of this representation is Fricke modular group Fr_d which is a subgroup of AL_d . Surprisingly [HLOYb] showed

Date: September 10, 2012.

that the index $[\text{AL}_d : \text{Fr}_d]$ is equal to the Fourier-Mukai number of X . Furthermore they predicts that all Atkin-Lehner involutions are obtained from Fourier-Mukai transformations $\Phi : D(Y) \rightarrow D(X)$ on X .

1.3. Our results. In our main theorem, Theorem 3.3, we show that Hosono-Lian-Oguiso-Yau's conjecture holds. To formulate our results transparently we introduce the notion of the groupoid \mathcal{FM}_X consisting of Fourier-Mukai transformations on X (See Definition 3.1). Moreover we construct an explicit correspondence between cosets of AL_d/Fr_d and Fourier-Mukai partners of X . Namely we have the surjective functor

$$\tilde{\rho} : \mathcal{FM}_X \rightarrow O^+(\mathcal{N}(X)),$$

where $O^+(\mathcal{N}(X))$ is the orientation preserving isometry group of $\mathcal{N}(X)$.

Now recall [HMS09, Corollary 3]: There is a surjection

$$\rho : \text{Aut}(D(X)) \rightarrow O_{\text{Hodge}}^+(H^*(X, \mathbb{Z}))$$

(See also Theorem 2.7). If we restrict $\tilde{\rho}$ to $\text{Aut}(D(X))$, this gives the representation ρ . Hence our theorem can be regarded as a slight generalization of [HMS09, Corollary 3].

Acknolegement. The author was partially supported by Grant-in-Aid for Scientific Research (S), No 22224001.

2. PRELIMINARIES

2.1. Induced morphisms on \mathbb{H} . To discuss the relation between Fourier-Mukai transformations and Atkin-Lehner modular group, we recall the representation of Fourier-Mukai transformations to $PSL_2(\mathbb{R})$, the automorphism group of the upper half plain \mathbb{H} .

We first consider the numerical Grothendieck group

$$\mathcal{N}(X) = H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z}).$$

The *Mukai paring* (or *Euler paring*) on $\mathcal{N}(X)$ is given by

$$\langle r \oplus c \oplus s, r' \oplus c' \oplus s' \rangle = cc' - rs' - sr'.$$

By the Hodge index theorem, the index of the Mukai paring is $(2, \rho(X))$.

For objects $E \in D(X)$ we put $v(E) = ch(E)\sqrt{td_X}$ and call it the *Mukai vector* of E . One can check that $v(E) = r \oplus c \oplus s \in \mathcal{N}(X)$ and see that $r = \text{rank } E$, $c = c_1(E)$ and $s = \chi(X, E) - \text{rank } E$ by using Riemann-Roch theorem.

Let $\mathfrak{D}^+(X)$ be one of the connected component

$$\{[v] \in \mathbb{P}(\mathcal{N}(X) \otimes \mathbb{C}) | v^2 = 0, v\bar{v} > 0\}$$

containing $[\exp(\sqrt{-1}\omega)]$ where ω is an ample divisor. As is well-known $\mathfrak{D}^+(X)$ is isomorphic to the tube domain $\text{NS}(X)_\mathbb{R} \times C^+(X)$ where $C^+(X)$ is the positive cone:

$$\text{NS}(X)_\mathbb{R} \times C^+(X) \ni (\beta, \gamma) \mapsto [\exp(\beta + \sqrt{-1}\gamma)] \in \mathfrak{D}^+(X).$$

We remark that if $\rho(X) = 1$, $\mathfrak{D}^+(X)$ is canonically isomorphic to the upper half plain \mathbb{H} :

$$\mathbb{H} \ni u + \sqrt{-1}v \mapsto [\exp((u + \sqrt{-1}v)L)] \in \mathfrak{D}^+(X),$$

where L is an ample basis of $\text{NS}(X)$.

Now suppose that X and Y are K3 surfaces with $\rho(X) = \rho(Y) = 1$ and $\Phi: D(Y) \rightarrow D(X)$ is an equivalence. We put the degree of X and Y by $2d$. Since Φ induces the orientation preserving isometry $\Phi^N: \mathcal{N}(Y) \rightarrow \mathcal{N}(X)$ by [HMS09, Theorem 2], we obtain the morphism

$$\Phi_*: \mathfrak{D}^+(Y) \rightarrow \mathfrak{D}^+(X).$$

Since both $\mathfrak{D}^+(Y)$ and $\mathfrak{D}^+(X)$ are \mathbb{H} , we obtain the automorphism on \mathbb{H} by using the canonical isomorphism:

$$\Phi_*(u_Y + \sqrt{-1}v_Y) = u_X + \sqrt{-1}v_X.$$

This automorphism was calculated by the author [Kaw12, Lemmas 3.1 and 3.2]. To explain these lemmas, we set the following:

$$v(\Phi(\mathcal{O}_y)) = r_X \oplus n_X L_X \oplus s_X \text{ and } v(\Phi^{-1}(\mathcal{O}_x)) = r_Y \oplus n_Y L_Y \oplus s_Y.$$

Here $x \in X$ and $y \in Y$ are closed points. Then Φ_* is given as follows:

Proposition 2.1 ([Kaw12, Lemmas 3.1 and 3.2]). *Let $\Phi: D(Y) \rightarrow D(X)$ be an equivalence between projective K3 surfaces with $\rho = 1$ and let Φ_* be the induced automorphism on \mathbb{H} .*

- (1) *We have $r = r_X = r_Y$. Moreover if $r_X = 0$, then*

$$\Phi_*(u_Y + \sqrt{-1}v_Y) = x_Y + m + \sqrt{-1}v_Y$$

for some $m \in \mathbb{Z}$.

- (2) *Suppose that $r \neq 0$. Then Φ_* is given by*

$$\Phi_*(u_Y + \sqrt{-1}v_Y) = \frac{1}{d|r|} \cdot \frac{-1}{(u_Y + \sqrt{-1}v_Y) - \frac{n_Y}{r}} + \frac{n_X}{r}.$$

Original proof is written in terms of Bridgeland stability conditions on X . So, for the convenience of readers we write the proof.

Proof. Recall $v(\mathcal{O}_x) = 0 \oplus 0 \oplus 1$. Then we see

$$-r_X = \langle v(\Phi(\mathcal{O}_y)), v(\mathcal{O}_x) \rangle = \langle v(\mathcal{O}_y), v(\Phi^{-1}(\mathcal{O}_x)) \rangle = -r_Y.$$

Thus we see $r_X = r_Y$. Moreover, if $r_X = 0$ then one can see that Y is isomorphic to X and that the equivalence Φ is numerically equivalent to $\otimes(mL_X)$ for some $m \in \mathbb{Z}$ via an isomorphism $f: Y \rightarrow X$ (The details are in [Kaw12, Lemma 3.1]).

The second assertion follows from [Kaw12, Lemma 3.2]. We recall the proof.

One can easily see

$$\Phi^H(\exp((u_Y + \sqrt{-1}v_Y)L_Y)) = \lambda \exp((u_X + \sqrt{-1}v_X)L_X)$$

for some $\lambda \in \mathbb{C}^*$.

Put $\beta_X + \sqrt{-1}\omega_X = u_X L_X + \sqrt{-1}v_X L_X$ (respectively Y). We can define the function $Z_{(\beta_X, \omega_X)} : \mathcal{N}(X) \rightarrow \mathbb{C}$, which is usually called a *central charge*:

$$\begin{aligned} Z_{(\beta_X, \omega_X)}(E) &:= \langle \exp(\beta_X + \sqrt{-1}\omega_X), v(E) \rangle \\ &= \frac{v(E)^2}{2r} + \frac{r}{2} \left(\omega_X + \sqrt{-1} \left(\frac{c}{r} - \beta_X \right) \right)^2, \end{aligned}$$

where $v(E) = r \oplus c \oplus s$. Then we see

$$\begin{aligned} \lambda &= -\langle \Phi^H(\exp(\beta_Y + \sqrt{-1}\omega_Y)), v(\mathcal{O}_x) \rangle \\ &= -\langle \exp(\beta_Y + \sqrt{-1}\omega_Y), v(\Phi^{-1}(\mathcal{O}_x)) \rangle \\ &= -Z_{(\beta_Y, \omega_Y)}(\Phi^{-1}(\mathcal{O}_x)), \end{aligned}$$

and

$$\begin{aligned} -1 &= \langle \exp(\beta_Y + \sqrt{-1}\omega_Y), v(\mathcal{O}_y) \rangle \\ &= \langle \Phi^H(\exp(\beta_Y + \sqrt{-1}\omega_Y)), v(\Phi(\mathcal{O}_y)) \rangle \\ &= \lambda \cdot Z_{(\beta_X, \omega_X)}(\Phi(\mathcal{O}_y)). \end{aligned}$$

Thus we have

$$1 = Z_{(\beta_Y, \omega_Y)}(\Phi^{-1}(\mathcal{O}_x)) \cdot Z_{(\beta_X, \omega_X)}(\Phi(\mathcal{O}_y)).$$

Since $v(\Phi(\mathcal{O}_y))^2 = v(\Phi^{-1}(\mathcal{O}_x))^2 = 0$, we have

$$Z_{(\beta_Y, \omega_Y)}(\Phi^{-1}(\mathcal{O}_x)) = \frac{r}{2} \left(v_Y + \sqrt{-1} \left(\frac{n_Y}{r} - u_Y \right) \right)^2 L_Y^2$$

and

$$Z_{(\beta_X, \omega_X)}(\Phi(\mathcal{O}_y)) = \frac{r}{2} \left(v_X + \sqrt{-1} \left(\frac{n_X}{r} - u_X \right) \right)^2 L_X^2.$$

Since $L_X^2 = L_Y^2 = 2d$ we see

$$(u_X - \frac{n_X}{r}) + \sqrt{-1}v_X = \frac{\pm 1}{d|r|} \cdot \frac{1}{(u_Y - \frac{n_Y}{r_Y}) + \sqrt{-1}v_Y}.$$

Since the left hand side is in the upper half plain \mathbb{H} , the imaginary part of the left hand side is positive. Hence we have

$$(u_X - \frac{n_X}{r}) + \sqrt{-1}v_X = \frac{-1}{d|r|} \cdot \frac{1}{(u_Y - \frac{n_Y}{r}) + \sqrt{-1}v_Y}.$$

Thus we have finished the proof. \square

2.2. Atkin-Lehner and Fricke involutions. In this section we recall the Atkin-Lehner involutions and Fricke involutions. As usual we put

$$\Gamma_0(d) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL_2(\mathbb{Z}) \mid \gamma \in d\mathbb{Z} \right\}.$$

For integers $s, d \in \mathbb{Z}$ we define the symbol $s||d$ by

$$(2.1) \quad s||d \iff s|d \text{ and } \gcd(s, \frac{d}{s}) = 1.$$

Suppose that $s||d$. We put

$$W_s = \left\{ \frac{1}{\sqrt{s}} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \in PSL_2(\mathbb{R}) \mid \gamma \in \frac{d}{s}\mathbb{Z} \text{ and } \delta \in s\mathbb{Z} \right\}.$$

W_s is also given as

$$W_s = \left\{ \begin{pmatrix} \alpha\sqrt{s} & \frac{\beta}{\sqrt{s}} \\ \gamma\frac{d}{s}\sqrt{s} & \delta\sqrt{s} \end{pmatrix} \in PSL_2(\mathbb{R}) \mid \alpha, \beta, \gamma \text{ and } \delta \in \mathbb{Z} \right\}.$$

In particular we see $W_1 = \Gamma_0(d)$.

For cosets W_s one can easily check the following:

Lemma 2.2 ([CN79]). *Each W_s is in the normalizer of $\Gamma_0(d)$ in $SL_2(\mathbb{R})$. In addition the coset classes W_s and $W_{s'}$ satisfies the following rule:*

$$W_s^2 = W_1, W_s W_{s'} = W_{s'} W_s = W_{s*s'},$$

$$\text{where } s * s' = \frac{ss'}{\gcd(s, s')^2}$$

Definition 2.3. We put

$$\text{AL}_d := \bigsqcup_{s||d} W_s \text{ and } \text{Fr}_d := W_1 \sqcup W_d.$$

We call AL_d and Fr_d respectively the *Atkin-Lehner modular group* and the *Fricke modular group*.

Remark 2.4. By the above lemma we see both sets AL_d and Fr_d have group structures. Moreover, AL_d is the abelian normalizer group of $\Gamma_0(d)$ in $PSL_2(\mathbb{R})$. Since $W_s W_d = W_{\frac{d}{s}}$, the coset decomposition of AL_d/Fr_d is given by

$$\text{AL}_d/\text{Fr}_d = \bigsqcup_{s||d} (W_s \sqcup W_{\frac{d}{s}}).$$

2.3. An explicit construction of Fourier-Mukai partners of X . In this subsection we recall the work of [HLOYa] which is an explicit construction of Fourier-Mukai partners of X with $\text{NS}(X) = \mathbb{Z}L$. Put $L^2 = 2d$ as usual.

We set the set P_d by

$$P_d = \{r \in \mathbb{N} \mid r||d\} / \sim$$

where $r_1 \sim r_2$ if and only if $r_1 = r_2$ or $r_1 = \frac{d}{r_2}$.

Theorem 2.5 ([HLOYa, Theorem 2.1]). *Let X be a projective K3 surface with $\text{NS}(X) = \mathbb{Z}L$. Put $L^2 = 2d$. There is a one to one correspondence between P_d and the set FM_X of isomorphic classes of Fourier-Mukai partners of X :*

$$P_d \ni r \mapsto M_L(r \oplus L \oplus \frac{d}{r}) \in \text{FM}_X.$$

Here $M_L(r \oplus L \oplus s)$ is the fine moduli space of μ_L -stable sheaves with Mukai vector $r \oplus L \oplus s$.

2.4. Lattices and modular groups. The aim of this subsection is to recall Dolgachev's theorem.

Let $N_d = \mathbb{Z}e_0 \oplus \mathbb{Z}\ell \oplus \mathbb{Z}e_4$ be the abstract lattice with the intersection matrix Σ where

$$\Sigma = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2d & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Let $O(N_d)$ be the orthogonal group of N_d :

$$O(N_d) = \{g \in GL_3(\mathbb{Z}) \mid {}^t g \Sigma g = \Sigma\}.$$

Put $O^+(N_d)$ be the subgroup consisting of $g \in O(N_d)$ which preserves the orientation of positive 2 plane in $N_{\mathbb{R}} = N_d \otimes_{\mathbb{Z}} \mathbb{R}$. Since the intersection form is non-degenerate, we see $N_d \subset N_d^{\vee} = \text{Hom}(N_d, \mathbb{Z})$ in $N_{\mathbb{R}}$. Hence $g \in O(N_d)$ induces the isometry on the discriminant lattice $A_{N_d} = N_d^{\vee}/N_d$ with respect to the natural quadratic form. We define $O(N_d)^*$ by the kernel of the morphism $O(N_d) \rightarrow O(A_{N_d})$ and define $O^+(N_d)^* = O^+(N_d) \cap O(N_d)^*$.

Now put $SO^+(N_d) = \{g \in O^+(N_d) \mid \det g = 1\}$. Then $SO^+(N_d)$ is isomorphic to $PSL(2, \mathbb{R})$ by the following morphism

$$R : PSL(2, \mathbb{R}) \rightarrow SO^+(N_d), \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \delta^2 & 2\gamma\delta & \frac{1}{d}\gamma^2 \\ \beta\delta & \alpha\delta + \beta\gamma & \frac{1}{d}\alpha\gamma \\ d\beta^2 & 2d\alpha\beta & \alpha^2 \end{pmatrix}.$$

Then we have the following sequence of morphisms:

$$q : O^+(N_d) \rightarrow O^+(N_d)/\pm \text{id}_{N_d} \xrightarrow{\sim} SO^+(N_d) \xrightarrow{R^{-1}} PSL(2, \mathbb{R}).$$

In this situation Dolgachev proves the following:

Theorem 2.6 ([Dol96, Theorem 7.1 and Remark 7.2]). *The image $q(O^+(N_d)^*)$ of $O^+(N_d)^*$ is the Fricke modular group Fr_d and the image $q(O^+(N_d))$ of $O^+(N_d)$ is the Atkin-Lehner modular group.*

2.5. Modular groups and autoequivalences on $D(X)$. By using Dolgachev's theorem and the theorem of [HMS09] (below), we discuss the relation between the Fricke modular group and the autoequivalence group $\text{Aut}(D(X))$.

Since X is a K3 surface, any autoequivalence $\Phi \in \text{Aut}(D(X))$ induces the Hodge isometry Φ^H of the integral cohomology ring $H^*(X, \mathbb{Z})$.

Theorem 2.7 ([HMS09, Corollary 3]). *Let X be a projective K3 surface (not necessary Picard rank 1). Then the morphism*

$$\rho : \text{Aut}(D(X)) \rightarrow O_{\text{Hodge}}(H^*(X, \mathbb{Z})), \Phi \mapsto \Phi^H$$

is surjective to the orientation preserving isometry group $O_{\text{Hodge}}^+(H^(X, \mathbb{Z}))$.*

In particular we obtain the isometry $\Phi^H|_{\mathcal{N}(X)}$ on the numerical Grothendieck group $\mathcal{N}(X)$. Suppose $\rho(X) = 1$ with $L_X^2 = 2d$. Since $\mathcal{N}(X)$ is canonically isomorphic to the abstract lattice N_d . Thus we have the morphism:

$$(2.2) \quad M : \text{Aut}(D(X)) \xrightarrow{\rho} O^+(H^*(X, \mathbb{Z})) \xrightarrow{|_{N_d}} O^+(N_d) \xrightarrow{q} PSL_2(\mathbb{R}).$$

By combining the above two theorems, we obtain the following proposition.

Proposition 2.8. *The morphism M is surjective to the Fricke modular group Fr_d .*

Proof. We first show $\text{Im}(M) \subset \text{Fr}_d$. By Theorem 2.6, it is enough to show that $\Phi^H|_{\mathcal{N}(X)}$ is in $\langle O^+(\mathcal{N}(X))^*, \pm \text{id} \rangle \subset O^+(N_d)$.

Since $\rho(X) = 1$, the restriction $\Phi^H|_{T(X)}$ to the transcendental lattice $T(X) (\subset H^*(X, \mathbb{Z}))$ is $\pm \text{id}_{T(X)}$ by the result of Oguiso [Ogu02, Lemma 4.1]. Moreover since a single shift [1] is in the kernel of M , we may assume $\Phi^H|_{T(X)} = \text{id}_{T(X)}$ by composing a single shift. We note that $\Phi^H|_{T(X)}$ induces the identity on the discriminant lattice of $T(X)$. Since the discriminant lattice $A_{T(X)} = T(X)^\vee/T(X)$ of $T(X)$ is canonically isomorphic to the discriminant lattice $A_{\mathcal{N}(X)}$ of $\mathcal{N}(X)$, $\Phi^H|_{\mathcal{N}(X)}$ is in $O^+(\mathcal{N}(X))^*$. Hence $\text{Im}(M)$ contained in Fr_d by Theorem 2.6.

Conversely we show $\text{Fr}_d \subset \text{Im}(M)$. Take an arbitrary $\varphi \in O^+(N(X))^*$. Then $\varphi \oplus \text{id}_{T(X)}$ extends to the isometry $\tilde{\varphi}$ on the hole lattice $H^*(X, \mathbb{Z})$. Since φ preserves the orientation of $\mathcal{N}(X)$, $\tilde{\varphi}$ also preserves the orientation of $H^*(X, \mathbb{Z})$. Since the natural representation $\text{Aut}(D(X)) \rightarrow O^+(H^*(X, \mathbb{Z}))$ is surjective by Theorem 2.7, there is an autoequivalence Φ such that $\Phi^H = \tilde{\varphi}$. \square

3. MAIN RESULT AND THE PROOF

We first remark that the set of all Fourier-Mukai transformations has naturally a groupoid structure. Namely we define the following:

Definition 3.1. Let M be a projective manifold. We define the groupoid $\mathcal{F}\mathcal{M}_M$ as follows:

- Objects of $\mathcal{F}\mathcal{M}_M$ consist of Fourier-Mukai partners of X :

$$\text{Ob}(\mathcal{F}\mathcal{M}_M) = \{W : \text{projective manifold} | \exists \Phi : D(W) \xrightarrow{\sim} D(M)\}.$$

- Morphisms in $\mathcal{F}\mathcal{M}_M$ are Fourier-Mukai transformations between them:

$$\text{Mor}_{\mathcal{F}\mathcal{M}_M}(W, W') = \{\Phi : D(W) \xrightarrow{\cong} D(W'), \text{FM transformations on } M\}$$

Since any Fourier-Mukai transformation gives $\Phi : D(W) \rightarrow D(W')$ a morphism of $\mathcal{F}\mathcal{M}_M$, we write as $\Phi \in \mathcal{F}\mathcal{M}_M$. We call $\mathcal{F}\mathcal{M}_M$ the *groupoid of Fourier-Mukai transformations* on M (or shortly *Fourier-Mukai groupoid*).

Now suppose that $M = X$ is a projective K3 surface with $\rho(X) = 1$. Let $\Phi : D(Y) \rightarrow D(Y')$ be in $\mathcal{F}\mathcal{M}_X$. Since the numerical Grothendieck

groups of Y and Y' are canonically isomorphic to the abstract lattice N_d , the equivalence Φ induces the orientation preserving isometry Φ^N on N_d . Namely we have the functor from the groupoid to the isometry group of N_d by using these canonically isomorphisms:

$$\rho' : \mathcal{F}\mathcal{M}_X \rightarrow O^+(N_d), \Phi \mapsto \Phi^N.$$

Remark 3.2. By composing the morphism $q : O^+(N_d) \rightarrow PSL_2(\mathbb{R})$, we obtain the following functor

$$M = q \circ \rho' : \mathcal{F}\mathcal{M}_X \rightarrow PSL_2(\mathbb{R}), \Phi \mapsto q(\Phi^N).$$

Since the restriction of $q \circ \rho'$ to $\text{Aut}(D(X))$ is the same as the group morphism $M : \text{Aut}(D(X)) \rightarrow PSL_2(\mathbb{R})$, we put $M = q \circ \rho'$ by abusing notations. By the definition of the functor M , we see $M(\Phi)$ is just the linear fractional transformation Φ_* given in Proposition 2.1

Theorem 3.3. *Let $\mathcal{F}\mathcal{M}_X$ be the Fourier-Mukai groupoid on a K3 surface X with $\rho(X) = 1$. We put $\text{NS}(X) = \mathbb{Z}L$ with $L^2 = 2d$.*

- (1) *The functor $M : \mathcal{F}\mathcal{M}_X \rightarrow \text{AL}_d$ is surjective. Namely for any $\varphi \in \text{AL}_d$, there exists a Fourier-Mukai transformation $\Phi : D(Y) \rightarrow D(X)$ in $\mathcal{F}\mathcal{M}_X$ such that $M(\Phi) = \varphi$.*
- (2) *For $\Phi : D(Y) \rightarrow D(Y') \in \mathcal{F}\mathcal{M}_X$, Y is isomorphic to Y' if and only if $M(\Phi) \in \text{Fr}_d$.*

Proof. Recall Proposition 2.8. By this proposition, it is enough to show that for any $s \parallel d$, there is a Fourier-Mukai transformation $\Phi : D(Y) \rightarrow D(X)$ such that $M(\Phi) \in W_s$.

For the integer s , we put $r = \frac{d}{s}$ and take an isotropic Mukai vector $v \in \mathcal{N}(X)$ as $v = r \oplus L_X \oplus s$. Then there exists the fine moduli spaces $M_L(r \oplus L \oplus s)$ of μ -stable sheaves with Mukai vector $v = r \oplus L \oplus s$ since $\gcd(r, L_X^2, s) = 1$. We put $Y = M_L(r \oplus L \oplus s)$ and let \mathcal{E} be the universal family of the moduli space. We claim that the Fourier-Mukai transformation $\Phi_{\mathcal{E}} : D(Y) \rightarrow D(X)$ satisfies $M(\Phi_{\mathcal{E}}) \in W_s$ where

$$\Phi_{\mathcal{E}}(-) : D(Y) \rightarrow D(X), \Phi_{\mathcal{E}}(-) = \mathbb{R}\pi_{X*}(\mathcal{E} \xrightarrow{\mathbb{L}} \pi_Y^*(-)).$$

Put $v(\Phi_{\mathcal{E}}^{-1}(\mathcal{O}_x)) = r \oplus nL_Y \oplus s'$. By Proposition 2.1 the linear fractional transformation $M(\Phi_{\mathcal{E}})$ is given by the following matrix:

$$M(\Phi_{\mathcal{E}}) = \begin{pmatrix} \sqrt{\frac{d}{r}} & -\sqrt{\frac{r}{d}} \frac{r+dn}{r^2} \\ r\sqrt{\frac{d}{r}} & -n\sqrt{\frac{d}{r}} \end{pmatrix}.$$

To prove our claim it is enough to show that $\frac{r+dn}{r^2}$ is an integer. To show this, we consider the inverse Fourier-Mukai transformation $\Phi_{\mathcal{E}}^{-1} : D(X) \rightarrow D(Y)$. Then the matrix $M(\Phi_{\mathcal{E}}^{-1})$ is given by

$$M(\Phi_{\mathcal{E}}^{-1}) = \begin{pmatrix} n\sqrt{\frac{d}{r}} & -\sqrt{\frac{r}{d}} \frac{r+dn}{r^2} \\ r\sqrt{\frac{d}{r}} & -\sqrt{\frac{d}{r}} \end{pmatrix}.$$

Since $(\Phi_{\mathcal{E}}^{-1})^N = \pm R \circ M(\Phi_{\mathcal{E}}^{-1}) \in O^+(N_d)$, all coefficient of the 3×3 matrix of $R \circ M(\Phi_{\mathcal{E}}^{-1})$ should be integers. By focusing $(2, 1)$ component of $R \circ M(\Phi_{\mathcal{E}}^{-1})$ we see that

$$\sqrt{\frac{r}{d}} \frac{r+dn}{r^2} \times \sqrt{\frac{d}{r}} = \frac{r+dn}{r^2}$$

is an integer. This gives the proof of the first assertion.

Now we prove the second assertion. Let $\Phi: D(Y_1) \rightarrow D(Y_2) \in \mathcal{F}\mathcal{M}_X$. By Theorem 2.5 we can assume $Y_i \xrightarrow{f_i} M_i = M_{L_X}(r_i \oplus L_X \oplus s_i)$ ($i = 1, 2$) with $r_i \mid d$. By using these isomorphisms we get the Fourier-Mukai transformation

$$\Phi' = f_{2*} \circ \Phi \circ f_{1*}^{-1}: D(M_1) \rightarrow D(M_2).$$

We note that $M(\Phi') = M(\Phi)$ since $M(f_{i*}) = \text{id}$ ($i = 1, 2$).

By the proof of the first assertion we see that there is an equivalence $\Psi_i: D(M_i) \rightarrow D(X)$ such that $M(\Phi_i) \in W_{\frac{d}{r_i}}$. Then we get the following commutative diagram:

$$\begin{array}{ccc} D(M_1) & \xrightarrow{\Phi'} & D(M_2) \\ \Psi_1 \downarrow & & \downarrow \Psi_2 \\ D(X) & \xrightarrow{\Psi_2 \cdot \Phi' \cdot \Psi_1^{-1}} & D(X) \end{array}$$

Since $\tilde{\Phi} = \Psi_2 \circ \Phi' \circ \Psi_1^{-1}$ is an autoequivalence, $M(\tilde{\Phi}) \in W_1 \sqcup W_d$. In particular by composing an equivalence $T \in \text{Aut}(D(X))$ so that $M(T) \in W_d$, we can assume that $M(\tilde{\Phi}) \in W_1$. Since $\Phi \in W_1 \sqcup W_d$, we have to consider two cases: If $\Phi \in W_1$ then we have

$$W_{\frac{d}{r_1}} W_1 W_{\frac{d}{r_2}} = W_{s_1} \cdot W_{s_2} = W_1.$$

Hence we see $s_1 = s_2$ and $r_1 = r_2$. Thus $Y_1 = Y_2$.

If $\Phi \in W_d$ then

$$W_{\frac{d}{r_1}} W_d W_{\frac{d}{r_2}} = W_{r_1} \cdot W_{s_2} = W_1.$$

Thus we see $r_1 = s_2$. Since $M(r \oplus L \oplus s) \cong M(s \oplus L \oplus r)$ by Theorem 2.5, we see $Y_1 \cong Y_2$. Thus we have proved the second assertion. \square

By combining Proposition 2.8, we obtain the following corollary.

Corollary 3.4. *The following functor is surjective:*

$$\tilde{\rho} = |_{N_d} \circ \rho: \mathcal{F}\mathcal{M}_X \ni \Phi \mapsto \Phi^N \in O^+(N_d).$$

Proof. Recall that the functor $M: \mathcal{F}\mathcal{M}_X \rightarrow PSL_2(\mathbb{R})$ factors through $O^+(N_d)$. Hence we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}\mathcal{M}_X & \xrightarrow{M} & \text{AL}_d \\ \searrow \tilde{\rho} & & \swarrow q \\ & O^+(N_d) & \end{array}$$

Since M and q are surjective by Theorems 2.6 and 3.3, we see $\tilde{\rho}$ is also surjective. \square

Remark 3.5. Corollary 3.4 can be regarded as the generalization of Theorem 2.7. Furthermore to generalize our result to arbitrary Picard rank cases, we have to find some canonical identification of numerical Grothendieck groups between Fourier-Mukai partners.

REFERENCES

- [BM01] T. Bridgeland and A. Maciocia, *Complex surfaces with equivalent derived categories*. Math. Z., **236** (2001), 677–697.
- [CN79] J. H. Conway and S. P. Norton, *Monstrous Moonshine*. Bull. London Math. Soc., **11** (1979), 308–339.
- [Dol96] I. V. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*, Algebraic Geometry, 4. J. Math. Sci., **81** (1996), 2599–2630.
- [Fav12] D. Favero, *Reconstruction and finiteness results for Fourier-Mukai partners*. Advances in Math., **230** (2012), 1955–1971.
- [HLOYa] S. Hosono, B. H. Lian, K. Oguiso and S-T. Yau, *Fourier-Mukai partners of a K3 surfaces of Picard number one*, in: Vector bundles and representation theory, Contemp. Math., 322, Amer. Math. Soc., Providence, RI, 2003, 43–55.
- [HLOYb] S. Hosono, B. H. Lian, K. Oguiso and S-T. Yau, *Autoequivalences of derived category of a K3 surface and monodromy transformations*. J. Algebraic Geom. **13** (2004), 513–545.
- [HMS09] D. Huybrechts, E. Macrì and P. Stellari, *Derived equivalences of K3 surfaces and orientation*. Duke Math. J. **149** (2009), 461–507.
- [Huy] D. Huybrechts. Fourier-Mukai transformations in Algebraic geometry. Oxford Science Publications.
- [Kaw02] Y. Kawamata, *D-Equivalence and K-Equivalence*. J. Diff. Geometry. **61** (2002), 147–171.
- [Kaw12] K. Kawatani, *A hyperbolic metric and stability conditions on K3 surfaces with $\rho = 1$* . preprint, arXiv:1204.1128.
- [Ogu02] K. Oguiso, *K3 surfaces via almost primes*. Math. Res. Lett. **9** (2002), 47–63.
- [Orl02] D. O. Orlov, *Derived categories of coherent sheaves on Abelian varieties and equivalence between them*. Izv. Math., **66** (2002), 569–594.

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